Automatica 50 (2014) 2051–2061

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Duality and network theory in passivity-based cooperative control ☆

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ARTICLE INFO

Article history:
Received 16 January 2013
Received in revised form
14 April 2014
Accepted 23 April 2014
Available online 21 June 2014

Keywords:
Cooperative control
Networks
Convex optimization

ABSTRACT

This paper presents a class of passivity-based cooperative control problems that have an explicit connection to convex network optimization problems. The new notion of maximal equilibrium independent passivity is introduced and it is shown that networks of systems possessing this property asymptotically approach the solutions of a dual pair of network optimization problems, namely an *optimal potential* and an *optimal flow* problem. This connection leads to an interpretation of the dynamic variables, such as system inputs and outputs, to variables in a network optimization framework, such as divergences and potentials, and reveals that several duality relations known in convex network optimization theory translate directly to passivity-based cooperative control problems. The presented results establish a strong and explicit connection between passivity-based cooperative control theory on the one side and network optimization theory on the other, and they provide a unifying framework for network analysis and optimal design. The results are illustrated on a nonlinear traffic dynamics model that is shown to be asymptotically clustering.

1. Introduction

One of the most profound concepts in mathematics is the notion of *duality*. This concept manifests itself across many mathematical disciplines, but perhaps the most elegant and complete notion of duality is the celebrated Lagrange duality in convex optimization (Boyd & Vandenberghe, 2003). One of the most complete expositions of this duality theory relates to a class of optimization problems over networks, generally known as *network optimization* (Rockafellar, 1998). In Rockafellar (1998), a unifying framework for network optimization was established, with the key elements being a pair of dual optimization problems: the *optimal flow problem* and the *optimal potential problem*.

The notion of duality also has a long history within the theory of control systems, see e.g. Balakrishnan and Vandenberghe (1995). A recent trend in modern control theory is the study of cooperative control problems amongst groups of dynamical systems that interact over an information exchange network. A fundamental goal for the analysis of these systems is to reveal the interplay between properties of the individual dynamic agents, the underlying network topology, and the interaction protocols that influence the functionality of the overall system (Mesbahi & Egerstedt, 2010). Amongst the numerous control theoretic approaches being pursued to define a general theory for networks of dynamical systems, *passivity* takes an outstanding role; see e.g., Bai, Arcak, and Wen (2011). The conceptual idea underlying passivity-based cooperative control is to separate the network analysis and synthesis into two layers. On the systems layer, each dynamical system is designed to have a certain input–output behavior, namely passivity. Then, the complete network can be analyzed by considering only the input–output behavior of the individual systems and the network topology describing their interconnections. This conceptual idea was pursued in Arcak (2007), where group coordination problems were investigated. Following this, passivity was used in Zelazo and Mesbahi (2010) to derive performance bounds on the input/output behavior of consensus-type networks. Passivity is also widely used in coordinated control of robotic systems (Chopra & Spong, 2006). The related concepts of incremental passivity and relaxed co-coercivity have been used to study synchronization problems in Scardovi, Arcak, and Sontag (2010) and Stan and Sepulchre (2007). Passivity was also used in the context of Port-Hamiltonian systems on graphs in Van der Schaft and Maschke (2013), and it has been used to study *clustering* in networks with saturated couplings (Bürger, Zelazo, & Allgöwer, 2011, 2012, 2013).

☆ The material in this paper was partially presented at the 4th IFAC Workshop on Distributed Estimation and Control in Networked Systems, September 25–26, 2013, Koblenz, Germany. This paper was recommended for publication in revised form by Associate Editor Yoshito Ohta under the direction of Editor Roberto Tempo.

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http://dx.doi.org/10.1016/j.automatica.2014.06.002

In Hines, Arcak, and Packard (2011), a new definition of passivity was introduced that serves the needs of networked systems. In particular, the notion of *equilibrium independent passivity* characterizes dynamical systems that are passive with respect to an arbitrary equilibrium point. Equilibrium independent passivity enables a convergence analysis of dynamical networks without computing the convergence point a priori. A similar passivity concept can be found in Jayawardhana, Ortega, Gracia-Canseco, and Castanos (2007).

The passivity-based cooperative control framework and the network optimization framework have many modeling similarities, as both rely on a certain matrix description of the underlying network (i.e., the incidence matrix). However, to the best of our knowledge, an explicit connection between these two research areas has not yet been established, and this motivates the main thesis of this work: *Does the cooperative control framework inherit any of the duality results found in network optimization*? We present in this paper a class of networks consisting of dynamical systems with certain passivity properties that are intimately related to the network optimization theory of Rockafellar (1998) and admit similar duality interpretations. Our results build an analytic bridge between cooperative control theory and network optimization theory.

The contributions are as follows. Building upon Hines et al. (2011), we introduce a refined version of equilibrium independent passivity, named *maximal equilibrium independent passivity*. The new definition is motivated by the fact that the original definition excludes some important system classes, such as integrators. Equipped with this new notion of passivity, we consider a cooperative control framework involving maximal equilibrium independent passive systems that aim to reach output agreement. First, necessary conditions for an output agreement solution to exist are derived. It is then shown that any steady-state configuration is *inverse optimal* in the sense that the corresponding input solves a certain optimal flow problem while the output solves the dual optimal potential problem. Exploiting this connection, certain results on the existence and uniqueness of output agreement solutions are derived. Following this, conditions on the couplings are derived that ensure the output agreement steady state is realized. The dynamic state and the output variable of the couplings are also shown to be inverse optimal with respect to a dual pair of network optimization problems. The inverse optimality and duality results are then generalized to a broader class of networks of maximal equilibrium independent passive systems. The general results are used to analyze a nonlinear traffic dynamic model that is shown to asymptotically exhibit a clustering behavior.

The remainder of the paper is organized as follows. The network optimization framework of Rockafellar (1998) is reviewed in Section 2. In Section 3 the dynamical network model is introduced, some results on passivity-based cooperative control are reviewed, and the new notion of maximal equilibrium independent passivity is introduced. The connection to network optimization theory in form of inverse optimality conditions are established in Section 4. The inverse optimality results are then generalized to networks of maximal equilibrium independent passive systems in Section 5. The theoretical results are illustrated on a nonlinear traffic dynamics model in Section 6. We offer some concluding remarks in Section 7.

Preliminaries

A function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be *strongly monotone* on $\mathcal{D}$ if there exists $\alpha > 0$ such that $(\phi(\eta) - \phi(\xi))^\top(\eta - \xi) \geq \alpha\|\eta - \xi\|^2$ for all $\eta, \xi \in \mathcal{D}$, and *co-coercive* on $\mathcal{D}$ if there exists $\gamma > 0$ such that $(\phi(\eta) - \phi(\xi))^\top(\eta - \xi) \geq \gamma\|\phi(\eta) - \phi(\xi)\|^2$ for all $\eta, \xi \in \mathcal{D}$, see, e.g., Zhu and Marcotte (1995). A function $\Phi : \mathbb{R}^q \mapsto \mathbb{R}$ is said to be *convex* on a convex set $\mathcal{D}$ if for any two points $\eta, \xi \in \mathcal{D}$ and for all $\lambda \in [0, 1]$, $\Phi(\lambda\eta + (1 - \lambda)\xi) \leq \lambda\Phi(\eta) + (1 - \lambda)\Phi(\xi)$. It is said to be *strictly convex* if the inequality holds strictly and *strongly convex* on $\mathcal{D}$ if there exists $\alpha > 0$ such that for any two points $\eta, \xi \in \mathcal{D}$, with $\eta \neq \xi$, and for all $\lambda \in [0, 1]$, $\Phi(\lambda\eta + (1 - \lambda)\xi) < \lambda\Phi(\eta) + (1-\lambda)\Phi(\xi) - \frac{1}{2}\lambda(1-\lambda)\alpha\|\eta - \xi\|^2$. The *convex conjugate* of a convex function Φ, denoted $\Phi^\star$, is defined as (Rockafellar, 1997)

$$\Phi^\star(\xi) = \sup_{\eta\in\mathcal{D}}\{\eta^\top\xi - \Phi(\eta)\} = -\inf_{\eta\in\mathcal{D}}\{\Phi(\eta) - \eta^\top\xi\}. \tag{1}$$

The definition of a convex conjugate implies that for all η and ξ it holds that $\Phi(\eta) + \Phi^\star(\xi) \geq \eta^\top\xi$. A vector g is said to be a *subgradient* of a function Φ at η if $\Phi(\eta') \geq \Phi(\eta) + g^\top(\eta' - \eta)$. The set of all subgradients of Φ at η is called the *subdifferential* of Φ at η and is denoted by $\partial\Phi(\eta)$. The multivalued mapping $\partial\Phi : \eta \to \partial\Phi(\eta)$ is called the subdifferential of Φ, see Rockafellar (1997). A special convex function we employ is the *indicator function*. Let $\mathbb{C}$ be a closed, convex set, the indicator function is defined as

$$I_{\mathbb{C}}(\eta) = \begin{cases} 0 & \text{if } \eta \in \mathbb{C} \\ +\infty & \text{if } \eta \notin \mathbb{C}. \end{cases}$$

We will also use the indicator function for points, e.g., $I_0(\eta)$ as the indicator function for $\mathbb{C} = \{0\}$.

Given a control system $\dot{x} = f(x, u)$ with state $x \in \mathbb{R}^p$ and input $u \in \mathbb{R}^q$ and a function $S(x)$ mapping $\mathbb{R}^p$ to $\mathbb{R}$, the *directional derivative* of S is denoted by $\dot{S} = \frac{\partial S}{\partial x}f(x, u)$.

2. Network optimization theory

The objective of this paper is to study passivity-based cooperative control in the context of *network optimization theory* (Rockafellar, 1998). A *network* is described by a *graph* $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ consisting of a finite set of *nodes*, $\mathbf{V} = \{v_1, \dots, v_{|\mathbf{V}|}\}$, and a finite set of *edges*, $\mathbf{E} = \{e_1, \dots, e_{|\mathbf{E}|}\}$, describing the incidence relation between pairs of nodes. Although we consider $\mathcal{G}$ in the cooperative control problem as an *undirected graph*, we assign to each edge an arbitrary orientation. The notation $e_k = (v_i, v_j) \in \mathbf{E} \subset \mathbf{V} \times \mathbf{V}$ indicates that v_i is the initial node of edge e_k and v_j is the terminal node. For simplicity, we will abbreviate this with $k = (i, j)$, and write $k \in \mathbf{E}$ and $i, j \in \mathbf{V}$.

The *incidence matrix* $E \in \mathbb{R}^{|\mathbf{V}|\times|\mathbf{E}|}$ of the graph $\mathcal{G}$ with arbitrary orientation, is a $\{0, \pm 1\}$ matrix with the rows and columns indexed by the nodes and edges of $\mathcal{G}$ such that $[E]_{ik}$ has value '+1' if node i is the head of edge k, '−1' if it is the tail, and '0' otherwise. This definition implies that for any graph, $\mathbb{1}^\top E = 0$, where $\mathbb{1} \in \mathbb{R}^{|\mathbf{V}|}$ is the vector of all ones. We refer to the *circulation space* of $\mathcal{G}$ as the null space $\mathcal{N}(E)$, and the *differential space* of $\mathcal{G}$ as the range space $\mathcal{R}(E^\top)$; see Rockafellar (1998). Additionally, we call $\mathcal{N}(E^\top)$ the *agreement space*. Note that $\mathcal{N}(E^\top) \perp \mathcal{R}(E)$ and $\mathcal{N}(E) \perp \mathcal{R}(E^\top)$.

We call a vector $\boldsymbol{\mu} = [\mu_1, \dots, \mu_{|\mathbf{E}|}]^\top \in \mathbb{R}^{|\mathbf{E}|}$ a *flow* of the network $\mathcal{G}$. An element of this vector, μ_k, is the *flux* of the edge $k \in \mathbf{E}$. The incidence matrix can be used to describe a type of conservation relationship between the flow of the network along the edges and the net in-flow (or out-flow) at each node in the network, termed the *divergence* of the network $\mathcal{G}$. The net flux entering a node must be equal to the net flux leaving the node. The divergence associated with the flow $\boldsymbol{\mu}$ is denoted by the vector $\mathbf{u} = [\mathrm{u}_1, \dots, \mathrm{u}_{|\mathbf{V}|}]^\top \in \mathbb{R}^{|\mathbf{V}|}$ and can be represented as[2]

$$\mathbf{u} + E\boldsymbol{\mu} = 0. \tag{2}$$

Borrowing from electrical circuit theory, we call the vector $\mathbf{y} \in \mathbb{R}^{|\mathbf{V}|}$ a *potential* of the network $\mathcal{G}$. To any edge $k = (i, j)$, one can associate the *potential difference* as $\zeta_k = \mathrm{y}_j - \mathrm{y}_i$; we also call this the *tension* of the edge k. The *tension vector* $\boldsymbol{\zeta} = [\zeta_1, \dots, \zeta_{|\mathbf{E}|}]^\top$, can be

[2] This condition is *Kirchhoff's Current Law*.

expressed as[3]

$$\boldsymbol{\zeta} = E^\top \mathbf{y}. \tag{3}$$

Flows and tensions are related to potentials and divergences by the *conversion formula* $\boldsymbol{\mu}^\top \boldsymbol{\zeta} = -\mathbf{y}^\top \mathbf{u}$.

The *optimal flow problem* attempts to optimize the flow and divergence in a network subject to the conservation constraint (2). Each edge is assigned a *flux cost* $C_k^{flux}(\mu_k)$, and each node is assigned a *divergence cost* $C_i^{div}(\mathrm{u}_i)$, i.e.,

$$\min_{\mathbf{u},\boldsymbol{\mu}} \sum_{i=1}^{|\mathbf{V}|} C_i^{div}(\mathrm{u}_i) + \sum_{k=1}^{|\mathbf{E}|} C_k^{flux}(\mu_k) \quad \text{s.t. } \mathbf{u} + E\boldsymbol{\mu} = 0. \tag{4}$$

The problem (4) admits a dual problem with a very characteristic structure.[4] The objective functions of the dual problem turn out to be the convex conjugates of the original cost functions, i.e.,

$$C_i^{pot}(\mathrm{y}_i) := C_i^{div,\star}(\mathrm{y}_i) = -\inf_{\tilde{\mathrm{u}}_i}\{C_i^{div}(\tilde{\mathrm{u}}_i) - \mathrm{y}_i\tilde{\mathrm{u}}_i\}$$

and $C_k^{ten}(\zeta_k) := C_k^{flux,\star}(\zeta_k)$. In the dual, the linear constraint $\boldsymbol{\zeta} = E^\top \mathbf{y}$ must hold, and the resulting problem is the *optimal potential problem*

$$\min_{\mathbf{y},\boldsymbol{\zeta}} \sum_{i=1}^{|\mathbf{V}|} C_i^{pot}(\mathrm{y}_i) + \sum_{k=1}^{|\mathbf{E}|} C_k^{ten}(\zeta_k), \quad \text{s.t. } \boldsymbol{\zeta} = E^\top \mathbf{y}. \tag{5}$$

We provide in the sequel an interpretation of cooperative control problems for a certain class of passive systems in the context of these dual network optimization problems.

3. Passivity-based cooperative control

The basic model involving networks of passive dynamical systems with diffusive couplings is now introduced. A new notion of passivity, called *maximal equilibrium independent passivity*, is presented and we demonstrate it to be a well-suited concept for cooperative control.

3.1. A canonical dynamic network model

Networks of dynamical systems defined on an undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ are considered where each node represents a single-input single-output (SISO) system,

$$\Sigma_i: \quad \begin{aligned} \dot{x}_i(t) &= f_i(x_i(t), u_i(t), \mathrm{w}_i), \\ y_i(t) &= h_i(x_i(t), u_i(t), \mathrm{w}_i), \quad i \in \mathbf{V}, \end{aligned} \tag{6}$$

with state $x_i(t) \in \mathbb{R}^{p_i}$, input $u_i(t) \in \mathbb{R}$, output $y_i(t) \in \mathbb{R}$ and constant external signal w_i. In the following, we adopt the notation $\boldsymbol{y}(t) = [y_1(t), \ldots, y_{|\mathbf{V}|}(t)]^\top$ and $\boldsymbol{u}(t) = [u_1(t), \ldots, u_{|\mathbf{V}|}(t)]^\top$ for the stacked output and input vectors. Similarly, we use $\boldsymbol{x}(t) \in \mathbb{R}^{\sum_{i=1}^{|\mathbf{V}|} p_i}$ for the stacked state vector, $\mathbf{w}$ for the external signals and write $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \mathbf{w})$, $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}, \mathbf{w})$ for the complete stacked dynamical system. To each edge $k \in \mathbf{E}$, connecting two nodes $i, j \in \mathbf{V}$, we associate the relative output $\zeta_k(t) = y_i(t) - y_j(t)$. The relative outputs can be defined with the incidence matrix as

$$\boldsymbol{\zeta}(t) = E^\top \boldsymbol{y}(t). \tag{7}$$

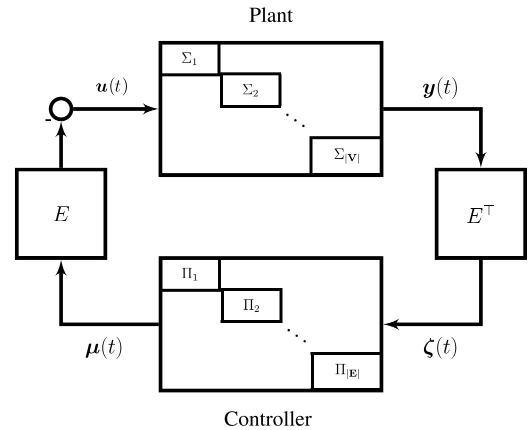


Fig. 1. Block-diagram of the canonical passivity-based cooperative control structure.

The relative outputs $\boldsymbol{\zeta}(t)$ drive dynamical systems placed on the edges of $\mathcal{G}$ that are of the form

$$\Pi_k: \quad \begin{aligned} \dot{\eta}_k(t) &= \zeta_k(t), \\ \mu_k(t) &= \psi_k(\eta_k(t)), \quad k \in \mathbf{E}. \end{aligned} \tag{8}$$

To account for the diffusive structure of the coupling controller, we rely throughout the paper on the standing assumption that the initial conditions satisfy $\boldsymbol{\eta}(0) \in \mathcal{R}(E^\top)$, which implies that $\boldsymbol{\eta}(t) \in \mathcal{R}(E^\top)$ for all times. The nonlinear functions ψ_k will be specified later on. The systems (8) will in the following be called controllers. The output of the controllers influence the incident systems as

$$\boldsymbol{u}(t) = -E\boldsymbol{\mu}(t). \tag{9}$$

The complete dynamical network (6)–(9) is illustrated in Fig. 1.

Remark 3.1. The model (6)–(9) includes the class of *diffusively coupled networks* of the form

$$\ddot{\chi}_i = f_i(\dot{\chi}_i) + \mathrm{w}_i + \sum_{j \in \mathcal{N}_i} \psi_{ij}(\chi_j - \chi_i),$$

where $\chi_i \in \mathbb{R}$, and $\mathcal{N}_i$ is the set of neighbors of node i in $\mathcal{G}$. If the nonlinear diffusive couplings $\psi_{ij}(\chi_j - \chi_i)$ are realized by odd functions and $\psi_{ij} = \psi_{ji}$, then the system can be represented in the form (6)–(9), with $x_i = \dot{\chi}_i$.

Remark 3.2. The model (6)–(9) is closely related to *Hamiltonian systems on graphs* (Van der Schaft & Maschke, 2013). Suppose there exists a Hamiltonian function $H: \mathbb{R}^{|\mathbf{E}|} \times \mathbb{R}^{|\mathbf{V}|} \to \mathbb{R}$, then a port-Hamiltonian system on a graph takes the form

$$\begin{bmatrix} \dot{\boldsymbol{\eta}}(t) \\ \dot{\boldsymbol{x}}(t) \end{bmatrix} = \begin{bmatrix} 0 & E^\top \\ -E & -D \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial \boldsymbol{\eta}}(\boldsymbol{\eta}(t), \boldsymbol{x}(t)) \\ \dfrac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{\eta}(t), \boldsymbol{x}(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} \mathrm{w}. \tag{10}$$

The matrix D is a positive semi-definite "damping" matrix. If D is a diagonal matrix, and $\frac{\partial H}{\partial \boldsymbol{\eta}}(\boldsymbol{\eta}(t), \boldsymbol{x}(t))$ and $\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{\eta}(t), \boldsymbol{x}(t))$ are solely functions of $\boldsymbol{\eta}(t)$ and $\boldsymbol{x}(t)$, respectively, then the model is in the form (6)–(9).

3.2. Passivity as a sufficient condition for convergence

A common theme in the existing literature is to exploit passivity properties for a convergence analysis. The convergence results can be traced back to well-known feedback theorems (Khalil, 2002), and we review a basic convergence result here. From here on we

[3] This condition is *Kirchoff's Voltage Law*.

[4] To form the dual problem, one can replace the divergence u_i and flow μ_k variables in the objective functions with artificial variables $\tilde{\mathrm{u}}_i$ and $\tilde{\mu}_k$, respectively, and introduce the artificial constraints $\mathrm{u}_i = \tilde{\mathrm{u}}_i$, $\mu_k = \tilde{\mu}_k$. These artificial constraints can be dualized with Lagrange multipliers y_i and ζ_k, respectively.

use the notational convention that italic letters denote dynamic variables, e.g., $y(t)$, and letters in normal font denote constant signals, e.g., y.

Assumption 3.3. There exist constant signals $\mathbf{u}, \mathbf{y}, \boldsymbol{\mu}, \boldsymbol{\zeta}$ such that $\mathbf{u} = -E\boldsymbol{\mu}$, $\boldsymbol{\zeta} = E^\top \mathbf{y}$ and

(i) each dynamic system (6) is output strictly passive with respect to u_i and y_i, i.e., there exists a positive semi-definite storage function $S_i(x_i(t))$ and a constant $\rho_i > 0$ such that

$$\dot{S}_i \leq -\rho_i \|y_i(t) - \mathrm{y}_i\|^2 + (y_i(t) - \mathrm{y}_i)(u_i(t) - \mathrm{u}_i); \tag{11}$$

(ii) each controller (8) is passive with respect to ζ_k and μ_k, i.e., there exists a positive semi-definite storage function $W_k(\eta_k(t))$ such that

$$\dot{W}_k \leq (\mu_k(t) - \mathrm{\mu}_k)(\zeta_k(t) - \mathrm{\zeta}_k).$$

Now, the basic convergence result follows directly.

Theorem 3.4 (*Convergence of Passive Networks*). *Consider the dynamical network (6)–(9) and suppose Assumption 3.3 holds, then the output variables $\boldsymbol{y}(t)$ converge to a constant steady-state value $\mathbf{y}$, i.e.,* $\lim_{t\to\infty} \boldsymbol{y}(t) \to \mathbf{y}$.

Proof. The passivity condition implies that

$$\begin{aligned}\sum_{i=1}^{|\mathbf{V}|} \dot{S}_i &\leq -\sum_{i=1}^{|\mathbf{V}|} \rho_i \|y_i(t) - \mathrm{y}_i\|^2 + (\boldsymbol{y}(t) - \mathbf{y})^\top (\boldsymbol{u}(t) - \mathbf{u}) \\ &= -\sum_{i=1}^{|\mathbf{V}|} \rho_i \|y_i(t) - \mathrm{y}_i\|^2 - (\boldsymbol{\zeta}(t) - \boldsymbol{\zeta})^\top (\boldsymbol{\mu}(t) - \boldsymbol{\mu}) \\ &\leq -\sum_{i=1}^{|\mathbf{V}|} \rho_i \|y_i(t) - \mathrm{y}_i\|^2 - \sum_{k=1}^{|\mathbf{E}|} \dot{W}_k.\end{aligned}$$

One can bring $\sum_{k=1}^{|\mathbf{E}|} \dot{W}_k$ to the left of the inequality and invoking Barbalat's lemma (Khalil, 2002) to conclude convergence, i.e., $\lim_{t\to\infty} \|\boldsymbol{y}(t) - \mathbf{y}\| \to 0$. $\square$

The appeal of this convergence result is that it decouples the dynamical systems layer and the network layer. Only the input–output behavior must be shown to be passive to conclude convergence of the overall network.

3.3. Equilibrium independent passivity

A critical aspect of the previous result relates to the assumption on the existence of the constant signals $\mathbf{u}, \mathbf{y}, \boldsymbol{\mu}, \boldsymbol{\zeta}$ that satisfy Assumption 3.3. The equilibrium configuration depends on the properties of all systems in the network and the desired passivity property cannot be verified locally. To overcome this issue, the concept of *equilibrium independent passivity* was introduced in Hines et al. (2011). Equilibrium independent passivity requires a system to be passive independent of the equilibrium point to which it is regulated.

Definition 3.5 (*Hines et al., 2011*). The system (6) is said to be *(output strictly) equilibrium independent passive* if there exists a set $\mathcal{U}_i \subset \mathbb{R}$ and a continuous function $k_{x,i}(\mathrm{u})$, defined on $\mathcal{U}_i$, such that (i) for any constant signal $\mathrm{u}_i \in \mathcal{U}_i$ the constant signal $\mathrm{x}_i = k_{x,i}(\mathrm{u}_i)$ is an equilibrium point of (6), i.e., $0 = f_i(\mathrm{x}_i, \mathrm{u}_i, \mathrm{w}_i)$, and (ii) the system is passive with respect to u_i and $\mathrm{y}_i = h_i(k_{x,i}(\mathrm{u}), \mathrm{u}_i, \mathrm{w}_i)$; that is, for each $\mathrm{u}_i \in \mathcal{U}_i$ there exists a storage function such that the inequality (11) holds (with $\rho_i \geq 0$ for equilibrium independent passivity and $\rho_i > 0$ for output-strictly equilibrium independent passivity).

The relevance of equilibrium independent passivity for the analysis of dynamical networks can be readily seen. If the systems (6) and (8) are output-strictly equilibrium independent passive and equilibrium independent passive, respectively, one has to verify only that an equilibrium trajectory exists in the respective sets to make the basic convergence proof of Theorem 3.4 applicable. The exact equilibrium point needs not be known.

One important implication of equilibrium independent passivity is that the equilibrium input–output map must be *monotone*, and even co-coercive, if the system is output-strictly equilibrium independent passive, see Hines et al. (2011).

3.4. Maximal equilibrium independent passivity

While equilibrium independent passivity turns out to be a useful concept for network analysis, the given definition excludes some important system classes. Consider for example a simple integrator, i.e., $\dot{x}_i(t) = u_i(t)$, $y_i(t) = x_i(t)$. It is well known that the integrator is passive with respect to $\mathcal{U}_i = \{0\}$ and any output value $\mathrm{y}_i \in \mathbb{R}$.[5] However, the equilibrium input–output map is not a (single-valued) *function* such that the integrator is not equilibrium independent passive as defined in Hines et al. (2011).

Motivated by this example, we propose here a refinement of equilibrium independent passivity. In particular, we do not require the equilibrium input–output maps $k_{\mathrm{y},i}$ to be *functions*, but instead allow them to be *relations* (or curves in $\mathbb{R}^2$). That is, $k_{\mathrm{y},i}$ is the set of all pairs $(\mathrm{u}_i, \mathrm{y}_i) \in \mathbb{R}^2$ that are equilibrium input–output relations. The domain of the relation is the set $\mathcal{U}_i$, i.e., $\operatorname{dom} k_{\mathrm{y},i} := \mathcal{U}_i$. We will sometimes write $k_{\mathrm{y},i}(\mathrm{u}_i)$ to denote the set of all y_i such that $(\mathrm{u}_i, \mathrm{y}_i) \in k_{\mathrm{y},i}$. This gives an interpretation of $k_{\mathrm{y},i}(\mathrm{u}_i)$ as *set-valued map*. For the integrator example described above, the equilibrium input–output relation is the vertical line through the origin, i.e., $k_{\mathrm{y},i} = \{(\mathrm{u}_i, \mathrm{y}_i) : \mathrm{u}_i = 0, \mathrm{y}_i \in \mathbb{R}\}$. For relations in $\mathbb{R}^2$ we review the concept of *maximal monotonicity*.

Definition 3.6 (*Rockafellar, 1998*). A relation $k_{\mathrm{y},i}$ is said to be *maximal monotone* if it cannot be embedded into a larger monotone relation. Equivalently, the relation $k_{\mathrm{y},i}$ is a maximal monotone relation if and only if

(i) for arbitrary $(\mathrm{u}_i, \mathrm{y}_i) \in k_{\mathrm{y},i}$ and $(\mathrm{u}'_i, \mathrm{y}'_i) \in k_{\mathrm{y},i}$ one has either $\mathrm{u}_i \leq \mathrm{u}'_i$ and $\mathrm{y}_i \leq \mathrm{y}'_i$, denoted by $(\mathrm{u}_i, \mathrm{y}_i) \leq (\mathrm{u}'_i, \mathrm{y}'_i)$, or $(\mathrm{u}_i, \mathrm{y}_i) \geq (\mathrm{u}'_i, \mathrm{y}'_i)$, and
(ii) for arbitrary $(\mathrm{u}_i, \mathrm{y}_i) \notin k_{\mathrm{y},i}$ there exists $(\mathrm{u}'_i, \mathrm{y}'_i) \in k_{\mathrm{y},i}$ such that neither $(\mathrm{u}_i, \mathrm{y}_i) \leq (\mathrm{u}'_i, \mathrm{y}'_i)$ nor $(\mathrm{u}_i, \mathrm{y}_i) \geq (\mathrm{u}'_i, \mathrm{y}'_i)$.

We refer to Rockafellar (1998) for a detailed treatment of maximal monotone relations. It is not difficult to see that the equilibrium input–output relation of the integrator system discussed above is maximal monotone. Based on this definition, a refined version of equilibrium independent passivity can be introduced. Please note that SISO systems are considered in this paper and the following definition applies only to SISO systems.

Definition 3.7 (*Maximal Equilibrium Independent Passivity*). A dynamical SISO system (6) is said to be *maximal equilibrium independent passive* if there exists a maximal monotone relation $k_{\mathrm{y},i} \subset \mathbb{R}^2$ such that for all $(\mathrm{u}_i, \mathrm{y}_i) \in k_{\mathrm{y},i}$ there exists a positive semi-definite storage function $S_i(x_i(t))$ satisfying

$$\dot{S}_i \leq (y_i(t) - \mathrm{y}_i)(u_i(t) - \mathrm{u}_i). \tag{12}$$

Furthermore, it is *output-strictly maximal equilibrium independent passive* if additionally there is a constant $\rho_i > 0$ such that

$$\dot{S}_i \leq -\rho_i \|y_i(t) - \mathrm{y}_i\|^2 + (y_i(t) - \mathrm{y}_i)(u_i(t) - \mathrm{u}_i). \tag{13}$$

[5] Passivity with respect to an arbitrary output $\mathrm{y}_i \in \mathbb{R}$ can be readily seen with the storage function $S_i(x_i(t)) = \frac{1}{2}(x_i(t) - \mathrm{y}_i)^2$.

The new notion of maximal equilibrium independent passivity is closely related to the definition of Hines et al. (2011). In fact, any equilibrium independent system with $\mathcal{U}_i = \mathbb{R}$ is also maximal equilibrium independent passive. This includes in particular affine dynamical systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + P\mathrm{w} \\ y(t) &= Cx(t) + Du(t) + G\mathrm{w}, \end{aligned} \tag{14}$$

that were shown in Hines et al. (2011) to be output strictly equilibrium independent passive if they are output-strictly passive in the classical sense for $\mathrm{w} = 0$ and if A is invertible. The equilibrium input–output relation is then the (single-valued) affine function (and thus a maximal monotone relation) $k_y(\mathrm{u}) = \left(-CA^{-1}B + D\right)\mathrm{u} + \left(-CA^{-1}P + G\right)\mathrm{w}$. Note that this is the dc-gain of the linear system plus the constant value determined by the exogenous inputs.

The two definitions also both include scalar nonlinear systems of the form

$$\dot{x}(t) = -f(x(t)) + u(t), \qquad y(t) = x(t), \tag{15}$$

with $x(t) \in \mathbb{R}, u(t) \in \mathbb{R}, y(t) \in \mathbb{R}$, for which $(x'(t) - x''(t))\left(f(x'(t)) - f(x''(t))\right) \geq \gamma(x'(t) - x''(t))^2$ for all $x', x'' \in \mathbb{R}$ and some constant $\gamma > 0$.

However, the integrator is the central example of a system that is included in the new definition of maximal equilibrium independent passivity, but not in the original one of Hines et al. (2011).

In the following section, networks of the structure (6)–(9) consisting of maximal equilibrium independent passive systems will be considered. It will be shown that these networks admit a certain *inverse optimality*.

4. Output agreement analysis

We now investigate the steady-state behavior of the dynamical network (6)–(9) and characterize an associated *inverse optimality* for these systems. To prepare the following discussion, we introduce some additional notation. We will write $\mathbf{k}_{\mathrm{y}}(\mathbf{u})$ for the stacked input–output relations, that is $\mathbf{y} \in \mathbf{k}_{\mathrm{y}}(\mathbf{u})$ means $\mathrm{y}_i \in k_{\mathrm{y},i}(\mathrm{u}_i)$ for all $i \in \mathbf{V}$. Similarly we will write $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_{|\mathbf{V}|}$ and $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_{|\mathbf{V}|}$ to indicate the domain and range of $\mathbf{k}_{\mathrm{y}}(\mathbf{u})$.

4.1. The plant level

The first observation we make is that a steady-state of the network (6)–(9) requires all systems to be in *output agreement*. Suppose that $\mathbf{x}$ and $\boldsymbol{\eta}$ are steady-state solutions of the network (6)–(9), and let $\mathbf{y}$ be the corresponding steady-state output, then

$$\mathbf{y} = \beta\mathbb{1},$$

for some $\beta \in \mathbb{R}$, called the *agreement value*. Output agreement follows from the steady-state condition $\dot{\boldsymbol{\eta}} = 0$, that requires $\mathbf{y} \in \mathcal{N}(E^\top)$. As $\mathcal{G}$ is connected, $\mathbf{y} \in \mathcal{N}(E^\top)$ is equivalent to $\mathbf{y} = \beta\mathbb{1}$ for some β.

The existence of an output agreement solution is related to the network equilibrium feasibility problem:

$$\begin{aligned} &\text{Find } \mathbf{u} \in \mathcal{R}(E), \quad \mathbf{y} \in \mathcal{N}(E^\top) \\ &\text{s.t. } \mathbf{y} \in \mathbf{k}_{\mathrm{y}}(\mathbf{u}). \end{aligned} \tag{16}$$

A necessary condition for the existence of an output agreement solution is now the following.

Lemma 4.1 (*Necessary Condition*)**.** *If the network (6)–(9) has a steady-state solution* $\mathbf{u}, \mathbf{y}$, *then this steady-state solves (16).*

Proof. The steady-state condition for the plant and for the controller require $\mathbf{y} \in \mathbf{k}_{\mathrm{y}}(\mathbf{u})$ and $\mathbf{y} \in \mathcal{N}(E^\top)$, respectively. Additionally, the interconnection (9) implies that $\boldsymbol{u}(t) \in \mathcal{R}(E)$, and consequently that $\mathbf{u} \in \mathcal{R}(E)$. □

To obtain further insights into the properties of an output agreement solution, we will next establish a connection to network optimization problems and show that certain duality relations hold. Therefore, some results relating maximal monotone relations and convex functions are recalled from Rockafellar (1998). A first observation is that one can extend any maximal monotone relation $k_{\mathrm{y},i} \subset \mathbb{R}^2$ with domain $\mathcal{U}_i$ to a maximal monotone relation on $\mathbb{R}$ by setting it to $-\infty$ for all u_i 'left' of $\mathcal{U}_i$ and $+\infty$ for all u_i 'right' of $\mathcal{U}_i$.[6] Now, we recall the following result of Rockafellar (1997, Thm. 24.9):

Theorem 4.2 (*Rockafellar, 1997*)**.** *The subdifferential for the closed proper convex functions on $\mathbb{R}$ are the maximal monotone relations from $\mathbb{R}$ to $\mathbb{R}$.*

Thus, one can associate to any maximal monotone relation, and consequently to any maximal equilibrium independent passive system, a closed proper convex function $K_i : \mathbb{R} \to \mathbb{R}$ that is unique up to an additive constant, such that

$$\partial K_i(\mathrm{u}_i) = k_{\mathrm{y},i}(\mathrm{u}_i) \quad \forall \mathrm{u}_i \in \mathcal{U}_i. \tag{17}$$

If $\mathcal{U}_i$ is not the complete $\mathbb{R}$ and the maximal monotone relation has been extended as described above, then $K_i(\mathrm{u}_i) = +\infty$ for all $\mathrm{u}_i \notin \mathcal{U}_i$. If the equilibrium input–output relation is a continuous single-valued function from $\mathbb{R}$ to $\mathbb{R}$ then $K_i(\mathrm{u}_i)$ is differentiable and $\nabla K_i(\mathrm{u}_i) = k_{\mathrm{y},i}(\mathrm{u}_i)$. We will call $K_i(\mathrm{u}_i)$ the *cost function* of the maximal equilibrium independent passive system i. Its convex conjugate, defined as in (1), i.e., $K_i^\star(\mathrm{y}_i) = \sup_{\mathrm{u}_i}\{\mathrm{y}_i\mathrm{u}_i - K_i(\mathrm{u}_i)\}$, is called the *potential function* of system i.

The steady states of the dynamical network of maximal equilibrium independent passive systems are intimately related to the following pair of dual network optimization problems.

Optimal flow problem: Consider the following *optimal flow problem*

$$\begin{aligned} &\min_{\mathbf{u},\boldsymbol{\mu}} \sum_{i=1}^{|\mathbf{V}|} K_i(\mathrm{u}_i) \\ &\text{s.t. } \mathbf{u} + E\boldsymbol{\mu} = 0. \end{aligned} \tag{OFP1}$$

This problem is of the form of (4). The costs on the divergences $\mathbf{u} \in \mathbb{R}^{|\mathbf{V}|}$ are the integral functions of the equilibrium input–output relations, i.e., $C_i^{div} = K_i$, and the flows $\boldsymbol{\mu} \in \mathbb{R}^{|\mathbf{E}|}$ on the edges are not penalized, i.e., $C_k^{flux} = 0$.

Optimal potential problem: Dual to the optimal flow problem, we define the following *optimal potential problem*

$$\begin{aligned} &\min_{\mathrm{y}_i} \sum_{i=1}^{|\mathbf{V}|} K_i^\star(\mathrm{y}_i), \\ &\text{s.t. } E^\top\mathbf{y} = 0. \end{aligned} \tag{OPP1}$$

This problem is in the form (5). The convex conjugates of the integral functions of the equilibrium input-to-output maps are the costs for the potential variables $\mathbf{y} \in \mathbb{R}^{|\mathbf{V}|}$ of the nodes, i.e., $C_i^{pot} = K_i^\star$. The constraint $E^\top\mathbf{y} = 0$ enforces a balancing of the potentials over the complete network. The problem can be written in the standard form (5), by choosing $C_k^{ten} = I_0$, i.e., the indicator function for the point zero. To simplify the presentation, we will use the short-hand notation $\mathbf{K}(\mathbf{u}) := \sum_{i=1}^{|\mathbf{V}|} K_i(\mathrm{u}_i)$ and $\mathbf{K}^\star(\mathbf{y}) := \sum_{i=1}^{|\mathbf{V}|} K_i^\star(\mathrm{y}_i)$.

The main result of this paper is that the output agreement steady-states in a network of maximal equilibrium independent passive systems admit an inverse optimality.

[6] Note that since $k_{\mathrm{y},i}$ is a maximal monotone relation, $\mathcal{U}_i$ is a connected interval on $\mathbb{R}$.

Theorem 4.3 (*Inverse Optimality of Output Agreement*)**.** *Suppose all node dynamics (6) are maximal equilibrium independent passive. If the network (6)–(9) has a steady-state solution* $\mathbf{u}, \mathbf{y}$, *then* (i) $\mathbf{u}$ *is an optimal solution to (OFP1),* (ii) $\mathbf{y}$ *is an optimal solution to (OPP1), and* (iii) *in the steady-state (OFP1) and (OPP1) have the same value with negative sign, i.e.,* $\sum_{i=1}^{|\mathbf{V}|} K_i(\mathrm{u}_i) + \sum_{i=1}^{|\mathbf{V}|} K_i^\star(\mathrm{y}_i) = 0$.

Proof. It is sufficient to show that the conclusions hold if the equilibrium problem (16) has a solution. If there is a solution $\mathbf{u}, \mathbf{y}$ to (16), then $\mathbf{u} \in \mathcal{R}(E) \cap \mathcal{U}$, while $\mathbf{y} \in \mathcal{N}(E^\top) \cup \mathcal{Y}$. Thus, both optimization problems have a feasible solution and are finite. Consider now the Lagrangian function of (OFP1) with multiplier $\tilde{\mathbf{y}}$, i.e.,

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\mu}, \tilde{\mathbf{y}}) = \sum_{i=1}^{|\mathbf{V}|} K_i(\mathbf{u}) - \tilde{\mathbf{y}}^\top \mathbf{u} + \tilde{\mathbf{y}}^\top E \boldsymbol{\mu}.$$

For $\mathbf{u}$ to be a solution to (OFP1), it is necessary and sufficient that $\tilde{\mathbf{y}} \in \partial \boldsymbol{K}(\mathbf{u})$ for the optimal multiplier $\tilde{\mathbf{y}}$. Thus, since $\partial \boldsymbol{K}(\mathbf{u}) = \mathbf{k}_{\mathrm{y}}(\mathbf{u})$, the multiplier satisfies $\tilde{\mathbf{y}} \in \mathbf{k}_{\mathrm{y}}(\mathbf{u})$.

To conclude that $\mathbf{u}$ is an optimal solution, it remains to show that the steady-state equilibrium trajectory $\mathbf{y}$ is an optimal multiplier, i.e., $\mathbf{y} = \tilde{\mathbf{y}}$. As $\mathbf{y}$ satisfies the equilibrium condition, it only remains to show that $\tilde{\mathbf{y}} = \mathcal{N}(E^\top)$. Let $s(\tilde{\mathbf{y}}) = \inf_{\mathbf{u},\boldsymbol{\mu}} \mathcal{L}(\mathbf{u}, \boldsymbol{\mu}, \tilde{\mathbf{y}})$. Now, if $\tilde{\mathrm{y}} \notin \mathcal{N}(E^\top)$ then $s(\tilde{\mathbf{y}})$ is unbounded below. For $\tilde{\mathbf{y}} \in \mathcal{N}(E^\top)$ it follows that $s(\tilde{\mathrm{y}}) = -\mathbf{K}^\star(\tilde{\mathbf{y}})$. Thus, the supremum problem is identical to (OPP1) with the negative objective function and both problems will have the same solution. Now, if the network equilibrium problem has a solution, then there must exist $\mathbf{u}$ and $\mathbf{y}$ satisfying the optimality conditions for the dual pair of optimization problems (OFP1) and (OPP1). Finally, as the steady-state solution is an optimal to both problems (OFP1) and (OPP1), it must be a saddle-point for the Lagrangian function, i.e., it must hold that

$$\sup_{\mathbf{y}} \inf_{\mathbf{u},\boldsymbol{\mu}} \mathcal{L}(\mathbf{u}, \boldsymbol{\mu}, \mathbf{y}) = \inf_{\mathbf{u},\boldsymbol{\mu}} \sup_{\mathbf{y}} \mathcal{L}(\mathbf{u}, \boldsymbol{\mu}, \mathbf{y}). \tag{18}$$

Let now $r(\mathbf{u}, \boldsymbol{\mu}) = \sup_{\mathbf{y}} \mathcal{L}(\mathbf{u}, \boldsymbol{\mu}, \mathbf{y})$. It follows that $r(\mathbf{u}, \boldsymbol{\mu}) = \mathbf{K}(\mathbf{u})$ if $\mathbf{u} + E\boldsymbol{\mu} = 0$ and $r(\mathbf{u}, \boldsymbol{\mu}) = +\infty$ otherwise. Additionally, we have already seen that $s(\mathrm{y}) = \inf_{\mathbf{u},\boldsymbol{\mu}} \mathcal{L}(\mathbf{u}, \boldsymbol{\mu}, \mathbf{y})$ is $s(\mathrm{y}) = -\mathbf{K}^\star(\mathbf{y})$ if $\mathbf{y} \in \mathcal{N}(E^\top)$ and $s(\mathrm{y}) = -\infty$ otherwise. For (18) to hold, the optimal solution $\mathbf{u} \in \mathcal{R}(E)$ and $\mathbf{y} \in \mathcal{N}(E^\top)$ must be such that $\boldsymbol{K}(\mathbf{u}) + \boldsymbol{K}^\star(\mathbf{y}) = 0$. As shown before, the steady-states of the dynamic network are optimal solutions to (OFP1) and (OPP1) and must therefore satisfy the previous equality. □

The connection between the agreement steady-state of the dynamical network and the dual pair of network optimization problems opens the way to use well-known tools form convex analysis for investigating the properties of output agreement steady-states.

Corollary 4.4 (*Existence*)**.** *Suppose all node dynamics are maximal equilibrium independent passive with* $\mathcal{U}_i = \mathbb{R}$ *and* $\mathcal{Y}_i = \mathbb{R}$, *then an output agreement steady-state exists.*

Proof. The dynamics of each node are assumed to be maximally equilibrium independent passive. Therefore, one can associate to each node the equilibrium input–output map $k_{\mathrm{y},i}$, and from Theorem 4.2 the closed proper convex functions K_i as well as the dual pair of network optimization problems (OFP1) and (OPP1). Since $\mathcal{U}_i = \mathbb{R}$ and $\mathcal{Y}_i = \mathbb{R}$, both optimization problems have a finite feasible solution and strong duality holds. The optimal primal–dual solution pair solves the equilibrium problem (16), and since $k_{\mathrm{y},i}$ is the equilibrium input–output map, it corresponds to an output agreement steady state. □

Corollary 4.5 (*Uniqueness*)**.** *Assume the dynamical systems (6) are maximal equilibrium independent passive with a nonempty* $\mathcal{U}_i$. *Furthermore, assume the equilibrium input–output functions* $k_{\mathrm{y},i}$ *are strongly monotone and satisfy* $\lim_{\ell\to\infty} |k_{\mathrm{y},i}(\mathrm{u}^\ell)| \to \infty$ *whenever* $\mathrm{u}^1, \mathrm{u}^2, \ldots$ *is a sequence in* $\mathcal{U}_i$ *converging to a boundary point of* $\mathcal{U}_i$. *Then there exists at most one pair* $(\mathbf{u}, \mathbf{y})$ *that can be a steady-state solution.*

Proof. From the assumptions follow that $K_i(\mathrm{u}_i)$ are differentiable and essentially smooth convex functions (see Rockafellar, 1997, p. 251). Thus, (OFP1) can have at most one solution. If such a solution exists, then the dual problem also has a solution. □

Corollary 4.6 (*Agreement Value*)**.** *Assume the same assumptions as for Corollary 4.5 hold. If an output agreement steady state exists, the agreement value* β *satisfies*

$$\sum_{i=1}^{|\mathbf{V}|} k_{\mathrm{y},i}^{-1}(\beta) = 0. \tag{19}$$

Proof. It follows from Theorem 26.1 in Rockafellar (1997) that $\nabla K_i^\star(\mathrm{y}_i) = k_{\mathrm{y},i}^{-1}(\mathrm{y}_i)$. Thus, after replacing $\mathbf{y}$ in (OPP1) with $\mathbf{y} = \beta\mathbb{1}$, the optimality condition of (OPP1) corresponds exactly to (19). □

4.2. The control level

It remains to investigate when the controller dynamics (8) can realize an output agreement steady state. In particular, in the steady-state configuration, the controller (8) must generate a signal $\boldsymbol{\mu}$ that corresponds to the desired control input. Suppose a solution $\mathbf{u}$ to (16) is known, then the controller must be such that the following static network equilibrium feasibility problem has a solution:

$$\begin{aligned} &\text{Find } \boldsymbol{\eta} \in \mathcal{R}(E^\top) \\ &\text{s.t.} \quad \mathbf{u} = -E\boldsymbol{\psi}(\boldsymbol{\eta}). \end{aligned} \tag{20}$$

Lemma 4.7 (*Necessary and Sufficient Condition*)**.** *The network (6)–(9) has a steady-state solution if and only if there exists a solution to (16) and (20).*

Proof. If the equilibrium problems have a solution $\mathbf{u}, \mathbf{y}, \boldsymbol{\eta}$, then $\mathbf{u}, \mathbf{y}, \boldsymbol{\mu} = \boldsymbol{\psi}(\boldsymbol{\eta})$ and $\boldsymbol{\zeta} = 0$ are a steady-state solution to (6)–(9). Any steady-state solution $\mathbf{u}, \mathbf{y}, \boldsymbol{\mu}, \boldsymbol{\zeta}$ of (6)–(9) solves the two equilibrium problems with $\boldsymbol{\mu} = \boldsymbol{\psi}(\boldsymbol{\eta})$. □

Please note that the two equilibrium problems (16) and (20) are not independent. However, if (16) has a unique solution, (20) has no influence on the solution of (16).

As the required steady-state input $\mathbf{u}$ is in general not known for the controller design, it seems appropriate to design the controller such that (20) is feasible for any $\mathbf{u} \in \mathcal{R}(E)$. Again, it will turn out that the feasibility of the network equilibrium problem is intimately related to maximal monotonicity. In particular, we show that (20) has a solution for all $\mathbf{u} \in \mathcal{R}(E)$ if ψ_k are *strongly monotone functions*.

Following this observation, we now assume that all ψ_k are strongly monotone functions. Then, one can associate to each edge $k \in \mathbf{E}$ a closed, proper strongly convex function $P_k : \mathbb{R} \to \mathbb{R}$ such that

$$\nabla P_k(\eta_k) = \psi_k(\eta_k). \tag{21}$$

Lemma 4.8. *Suppose the functions* ψ_k *are strongly monotone, then the controller dynamics (8) are maximal equilibrium independent passive.*

Proof. The equilibrium input set for the controller dynamics is solely $\zeta_k = 0$. However, the dynamics (8) is passive with respect to the input $\zeta_k = 0$ and any output $\mu_k \in \mathbb{R}$. To see this, consider the storage function

$$W_k(\eta_k(t), \mathrm{\eta}_k) = P_k(\eta_k(t)) - P_k(\mathrm{\eta}_k) - \nabla P_k(\mathrm{\eta}_k)(\eta_k(t) - \mathrm{\eta}_k),$$

where η_k is such that $\mu_k = \nabla P_k(\eta_k)$. From strict convexity of P_k follows directly that W_k is a positive definite function. Now, maximal passivity follows immediately from

$$\begin{aligned}\dot{W}_k &= (\nabla P_k(\eta_k(t)) - \nabla P_k(\eta_k))\zeta_k(t)\\ &= (\mu_k(t) - \mu_k)(\zeta_k(t) - \zeta_k),\end{aligned}$$

where we used that $\zeta_k = 0$. □

It will be shown next that the strong monotonicity of ψ_k ensures the existence of an output agreement steady-state solution and that the steady-state solution has additional inverse optimality properties. To see this, consider the following pair of dual network optimization problems.

Optimal potential problem: Let some $\mathbf{u} = [\mathrm{u}_1, \ldots, \mathrm{u}_{|\mathbf{V}|}]^\top \in \mathcal{R}(E)$ be given. Consider the following *optimal potential problem*

$$\min_{\boldsymbol{\eta}, \mathbf{v}} \sum_{k=1}^{|\mathbf{E}|} P_k(\eta_k) + \sum_{i=1}^{|\mathbf{V}|} \mathrm{u}_i \mathrm{v}_i, \quad \text{s.t. } \boldsymbol{\eta} = E^\top \mathbf{v}. \tag{OPP2}$$

By its structure, (OPP2) is an optimal potential problem as defined in (5). The potential vector $\mathbf{v}$ is associated to the linear cost defined by $\mathbf{u}$, while the tension variables $\boldsymbol{\eta}$ are associated to the integral functions of the coupling nonlinearities.

Optimal flow problem: The dual problem to (OPP2) is the following *optimal flow problem*

$$\min_{\boldsymbol{\mu}} \sum_{k=1}^{|\mathbf{E}|} P_k^\star(\mu_k) \quad \text{s.t. } \mathbf{u} + E\boldsymbol{\mu} = 0, \tag{OFP2}$$

where $P_k^\star$ is the convex conjugates of P_k, and $\mathbf{u} \in \mathcal{R}(E)$ is a given constant vector. The problem is in compliance with the standard form of optimal flow problems (4), as one can introduce artificial divergence variables and add as a cost function the indicator function for the point $\mathbf{u}$.

Theorem 4.9 (*Controller Realization*)**.** *Suppose the dynamical network nodes (6) are such that the necessary conditions of Theorem 4.3 are satisfied and the controller dynamics (8) are such that all ψ_k are strongly monotone. Then the network (6)–(9) has an output agreement steady-state solution. Furthermore, let $\boldsymbol{\eta}$ be the steady-state of the controller in output agreement, then* (i) *$\boldsymbol{\eta}$ is an optimal solution to (OPP2),* (ii) *$\boldsymbol{\mu} = \boldsymbol{\psi}(\boldsymbol{\eta})$ is an optimal solution to (OFP2),* (iii) *and $\sum_{k=1}^{|\mathbf{E}|} P_k^\star(\mu_k) + \sum_{k=1}^{|\mathbf{E}|} P_k(\eta_k) = \boldsymbol{\mu}^\top \boldsymbol{\eta}$.*

Proof. To prove the first claim, it is sufficient to show that for any $\mathbf{u} \in \mathcal{R}(E)$ the equilibrium problem (20) has a solution $\boldsymbol{\eta}$. At first we note that if ψ_k are strongly monotone, then P_k are strongly convex and are defined on $\mathbb{R}$. Now, note that the cost function of (OPP2) can be represented as a function of $\boldsymbol{\eta}$ only, since $\mathbf{u}^\top \mathbf{v} = -\boldsymbol{\mu}^\top \boldsymbol{\eta}$, for some $\boldsymbol{\mu}$ satisfying $\mathbf{u} = -E\boldsymbol{\mu}$, and thus, (OPP2) can be equivalently represented as the problem of minimizing a strongly convex cost function with effective domain $\mathbb{R}^{|\mathbf{E}|}$ over a linear subspace. Thus, (OPP2) has a unique solution $\boldsymbol{\eta}$ for all $\mathbf{u} \in \mathcal{R}(E)$. To prove the first claim, it remains to connect the solution of (OPP2) to the equilibrium condition (20). Any solution $\boldsymbol{\eta} = E^\top \mathbf{v}$ in (OPP2) must satisfy the first-order optimality condition

$$E\nabla \mathbf{P}(E^\top \mathbf{v}) + \mathbf{u} = 0,$$

where we use the short-hand notation $\mathbf{P} = \sum_{k=1}^{|\mathbf{E}|} P_k$. Since $\nabla \mathbf{P} = \boldsymbol{\psi}$, the optimal solution $\boldsymbol{\eta} = E^\top \mathbf{v}$ to (OPP2) solves explicitly the equilibrium condition (20), proving the first claim.

Now, to prove the remaining statements of the theorem, we consider the Lagrangian of (OPP2), i.e.,

$$\mathcal{L}(\mathbf{v}, \boldsymbol{\eta}, \tilde{\boldsymbol{\mu}}) = \sum_{k=1}^{|\mathbf{E}|} P_k(\eta_k) + \sum_{i=1}^{|\mathbf{V}|} \mathrm{u}_i \mathrm{v}_i + \tilde{\boldsymbol{\mu}}^\top(-\boldsymbol{\eta} + E^\top \mathbf{v}),$$

with multiplier $\tilde{\boldsymbol{\mu}}$. Define now the dual function as $s(\tilde{\boldsymbol{\mu}}) = \inf_{\mathbf{v},\boldsymbol{\eta}} \mathcal{L}(\mathbf{v}, \boldsymbol{\eta}, \tilde{\boldsymbol{\mu}})$. Clearly, $s(\tilde{\boldsymbol{\mu}}) = -\infty$ if $E\tilde{\boldsymbol{\mu}} + \mathbf{u} \neq 0$, and otherwise $s(\tilde{\boldsymbol{\mu}}) = -\mathbf{P}^\star(\tilde{\boldsymbol{\mu}})$. Thus, the dual problem $\sup s(\tilde{\mu})$ is equivalent to (OFP2) and the dual solution $\tilde{\boldsymbol{\mu}}$ is in fact the optimal solution to (OFP2). Together with the first order optimality condition this implies that $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}} = \nabla \mathbf{P}(\boldsymbol{\eta}) = \boldsymbol{\psi}(\boldsymbol{\eta})$. The last statement, i.e., the strong duality, follows since it must hold that

$$\sup_{\tilde{\boldsymbol{\mu}}} \inf_{\mathbf{v},\boldsymbol{\eta}} \mathcal{L}(\mathbf{v}, \boldsymbol{\eta}, \tilde{\boldsymbol{\mu}}) = \inf_{\mathbf{v},\boldsymbol{\eta}} \sup_{\tilde{\boldsymbol{\mu}}} \mathcal{L}(\mathbf{v}, \boldsymbol{\eta}, \tilde{\boldsymbol{\mu}}).$$

This implies that $\sup_{\tilde{\boldsymbol{\mu}}} s(\tilde{\mu})$ must take the same optimal value as (OPP2). The statement follows now since $\sup_{\tilde{\boldsymbol{\mu}}} s(\tilde{\mu})$ has the same value as (OFP2) with negative sign, and $\mathbf{u}^\top \mathbf{v} = -\boldsymbol{\mu}^\top \boldsymbol{\eta}$, where $\boldsymbol{\mu}$ is the optimal solution to (OFP2). □

The two optimization problems provide, on the one hand, explicit statements about the feasibility of the steady state independent of the required $\mathbf{u}$, and, on the other hand, additional duality relations. The internal state of the controller (8), $\boldsymbol{\eta}(t)$, can be understood as *tensions*, while the output of the controller, $\boldsymbol{\mu}(t)$, can be understood as the corresponding dual *flows*.

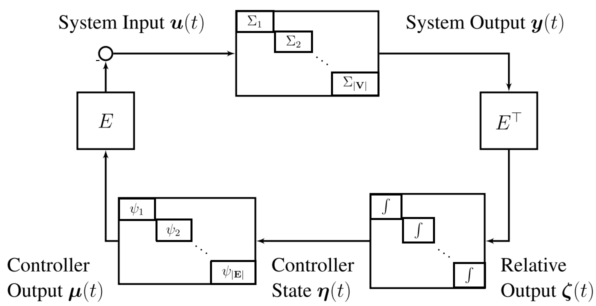
4.3. The closed-loop perspective

Having established conditions that ensure the existence and the optimality properties of an output agreement steady-state solution, it remains to prove convergence.

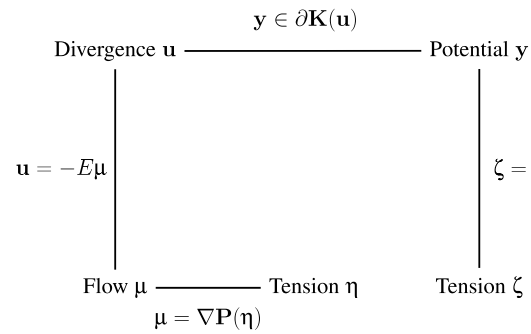
Theorem 4.10 (*Output Agreement*)**.** *Consider the dynamical network (6)–(9) and suppose that the nodes (6) are all output-strictly maximal equilibrium independent passive with $\mathcal{U}_i = \mathbb{R}$ and $\mathcal{Y}_i = \mathbb{R}$ and all coupling nonlinearities ψ_k are strongly monotone. Then there exist $\mathbf{u}$, $\mathbf{y}$, $\boldsymbol{\eta}$, and $\boldsymbol{\mu}$ being optimal solutions to (OFP1), (OPP1), (OPP2) and (OFP2), such that $\lim_{t\to\infty} \boldsymbol{u}(t) \to \mathbf{u}$, $\lim_{t\to\infty} \boldsymbol{y}(t) \to \mathbf{y}$, $\lim_{t\to\infty} \boldsymbol{\eta}(t) \to \boldsymbol{\eta}$, and $\lim_{t\to\infty} \boldsymbol{\mu}(t) \to \boldsymbol{\mu}$. In particular, the dynamical network converges to output agreement, i.e., $\lim_{t\to\infty} \boldsymbol{y}(t) \to \beta\mathbb{1}$.*

Proof. The assumptions ensure that the four network optimization problems (OFP1), (OPP1), (OPP2) and (OFP2) have an optimal solution. Thus, a steady-state solution exists. Output-strictly maximal equilibrium independent passivity of the node dynamics ensures that for all $i \in \mathbf{V}$ there exists a storage function S_i such that $\dot{S}_i \leq -\rho_i \|y_i(t) - \mathrm{y}_i\|^2 + (y_i(t) - \mathrm{y}_i)(u_i(t) - \mathrm{u}_i)$. Additionally, maximal equilibrium independent passivity of the controller dynamics ensures that for all $k \in \mathbf{E}$ there exists a storage function W_k satisfying $\dot{W}_k \leq (\mu_k(t) - \mu_k)(\zeta_k(t) - \zeta_k)$. Thus, the basic convergence result of Theorem 3.4 applies directly, proving convergence of the output trajectories, i.e., $\lim_{t\to\infty} \boldsymbol{y}(t) \to \mathbf{y}$. Since $\mathbf{y} \in \mathbf{k}_y(\mathbf{u})$, it follows that $\boldsymbol{u}(t)$ must converge to $\mathbf{u}$. The convergence of $\boldsymbol{\mu}(t)$ and $\boldsymbol{\zeta}(t)$ to $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$, respectively, follows immediately. □

We can summarize the results of this section as follows. All signals of the dynamical network (6)–(9) have static counterparts in the network optimization theory framework. The static counterparts of the outputs $\boldsymbol{y}(t)$ are the solutions $\mathbf{y}$ of the optimal potential problem (OPP1), while the corresponding dual variables, i.e., divergence variables in (OFP1), $\mathbf{u}$, are the static counterparts to the control inputs $\boldsymbol{u}(t)$. The controller state $\boldsymbol{\eta}(t)$ and the output $\boldsymbol{\mu}(t)$ have the tension and flow variables of (OPP2) and (OFP2), respectively, as their static counterparts. We visualize the connection between



(a) Signals of the closed-loop dynamical system.



(b) Variables of the network theoretic framework.

Fig. 2. The block diagram of the closed loop system (a) and the abstracted illustration of the network variables (b).

Table 1
Relation between variables involved in the dynamical system and their static counterparts.

Dynamic signal		Network variable		Relation	Cost function	Optimization problem
$\boldsymbol{y}(t)$	System output	$\mathbf{y}$	Potential	$\mathbf{y} = \mathbf{k}_y(\mathbf{u})$	$K_i^\star(y_i)$	(OPP1)
$\boldsymbol{\zeta}(t)$	Relative output	$\boldsymbol{\zeta}$	Tension	$\boldsymbol{\zeta} = E^\top \mathbf{y}$	$I_0(\zeta_k)$	(OPP1)
$\boldsymbol{u}(t)$	System input	$\mathbf{u}$	Divergence	$\mathbf{u} = \mathbf{k}_y{}^{-1}(\mathbf{y})$	$K_i(u_i)$	(OFP1)
$\boldsymbol{\mu}(t)$	Controller output	$\boldsymbol{\mu}$	Flow	$\mathbf{u} + E\boldsymbol{\mu} = 0$	$P_k^\star(\mu_k)$	(OFP2)
$\boldsymbol{v}(t)$	–	$\mathbf{v}$	Potential	$\boldsymbol{\eta} = E^\top \mathbf{v}$	$u_k v_k$	(OPP2)
$\boldsymbol{\eta}(t)$	Controller state	$\boldsymbol{\eta}$	Tension	$\boldsymbol{\mu} = \boldsymbol{\psi}(\boldsymbol{\eta})$	$P_k(\eta_k)$	(OPP2)

the dynamic variables of the closed-loop system and the static network variables in Fig. 2. A summary of all variables involved in the output agreement problem together with their static counterparts is provided in Table 1. For the sake of completeness, we include also the dynamic variable $\boldsymbol{v}(t)$, which corresponds to the potential variables $\mathbf{v}$ of (OPP2) and can be defined as $\boldsymbol{\eta}(t) = E\boldsymbol{v}(t)$.

5. A general dynamic network analysis framework

The full potential of the established duality framework can be seen if more general networks of maximal equilibrium independent passive systems are considered. We will generalize the previous results now for controllers (8) that are arbitrary maximal equilibrium independent passive systems. In particular, we assume now that the controllers (8) are replaced by dynamical systems of the form

$$\Pi_k: \quad \begin{aligned} \dot{\eta}_k &= \phi_k(\eta_k, \zeta_k) \\ \mu_k &= \psi_k(\eta_k, \zeta_k), \quad k \in \mathbf{E}. \end{aligned} \tag{22}$$

Assumption 5.1. The controllers (22) are maximal equilibrium independent passive with input set $\mathcal{Z}_k$, output set $\mathcal{M}_k$, and maximal monotone input–output relation $\gamma_k \subset \mathbb{R}^2$.

To each of the dynamics (22) one can associate now a closed, proper, convex function $\Gamma_k : \mathbb{R} \to \mathbb{R}$ such that

$$\partial \Gamma_k = \gamma_k . \tag{23}$$

Now, the formalism developed in the previous section can be generalized as the asymptotic behavior of the network (6), (7), (22), (9) and can be related to the following pair of dual network optimization problems.

Generalized optimal flow problem: Consider the following *optimal flow problem*

$$\min_{\mathbf{u},\boldsymbol{\mu}} \sum_{i=1}^{|\mathbf{V}|} K_i(u_i) + \sum_{k=1}^{|\mathbf{E}|} \Gamma_k^\star(\mu_k) \tag{GOFP}$$

s.t. $\mathbf{u} + E\boldsymbol{\mu} = 0$,

where $\Gamma_k^\star$ denotes the convex conjugate of Γ_k. This is a generalized version of (OFP1). Still the divergence $\mathbf{u}$ are associated to the cost functions defined by the integral of the nodes input–output relations. However, now the cost function $\Gamma_k^\star$ is associated to the flow variables μ_k.

Generalized optimal potential problem: Dual to the generalized optimal flow problem, we also define the generalized optimal potential problem as

$$\min_{\mathbf{y},\boldsymbol{\zeta}} \sum_{i=1}^{|\mathbf{V}|} K_i^\star(y_i) + \sum_{k=1}^{|\mathbf{E}|} \Gamma_k(\zeta_k) \tag{GOPP}$$

s.t. $\boldsymbol{\zeta} = E^\top \mathbf{y}$.

In contrast to (OPP1), this problem does not necessarily force the potential differences, i.e., the tensions, to be zero, but penalizes them with the general cost functions Γ_k.

The general network optimization problems (GOFP) and (GOPP) are related to the asymptotic behavior of the network of maximal equilibrium independent passive systems.

Theorem 5.2 (*Generalized Network Convergence Theorem*). *Consider the dynamical network (6), (7), (22) and (9). Assume all node dynamics (6) are output strictly maximal equilibrium independent passive and all controller dynamics (22) are maximal equilibrium independent passive, and the two network optimization problems (GOFP), (GOPP) have a feasible solution. Then there exists constant vectors $\mathbf{u}, \boldsymbol{\mu}$ solving (GOFP), and $\mathbf{y}, \boldsymbol{\zeta}$ solving (GOPP), such that $\lim_{t\to\infty} \boldsymbol{u}(t) \to \mathbf{u}$, $\lim_{t\to\infty} \boldsymbol{\mu}(t) \to \boldsymbol{\mu}$, $\lim_{t\to\infty} \boldsymbol{y}(t) \to \mathbf{y}$, and $\lim_{t\to\infty} \boldsymbol{\zeta}(t) \to \boldsymbol{\zeta}$.*

Proof. First, we show that if the two network optimization problems have a feasible solution, this solution represents an equilibrium for the dynamical network. Consider again the Lagrangian function of (GOFP) with Lagrange multiplier $\tilde{\mathbf{y}}$, i.e.,

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\mu}, \tilde{\mathbf{y}}) = \sum_{i=1}^{|\mathbf{V}|} K_i(u_i) + \sum_{k=1}^{|\mathbf{E}|} \Gamma_k^\star(\mu_k) + \tilde{\mathbf{y}}^\top(-\mathbf{u} - E\boldsymbol{\mu}).$$

Define now $\tilde{\boldsymbol{\zeta}} = E^\top \tilde{\mathbf{y}}$. If (GOFP) has an optimal solution, this solution satisfies the optimality conditions

$$\begin{aligned} &\partial \mathbf{K}_i(\mathbf{u}) - \tilde{\mathbf{y}} \in 0, \qquad \partial \boldsymbol{\Gamma}^\star(\boldsymbol{\mu}) - \tilde{\boldsymbol{\zeta}} \in 0 \\ &\mathbf{u} + E\boldsymbol{\mu} = 0, \qquad \tilde{\boldsymbol{\zeta}} = E^\top \tilde{\mathbf{y}}, \end{aligned} \tag{24}$$

where we use the notation $\mathbf{\Gamma}^\star(\boldsymbol{\mu}) = \sum_{k=1}^{|\mathbf{E}|} \Gamma_k^\star(\mu_k)$. Since $\mathbf{\Gamma}(\boldsymbol{\zeta}) = \sum_{k=1}^{|\mathbf{E}|} \Gamma_k(\zeta_k)$ is a closed convex function it follows from the inversion of the subgradients (i.e., Rockafellar (1997, Thm. 23.5)) that $\partial \mathbf{\Gamma}^\star(\boldsymbol{\mu})$ is equivalent to $\boldsymbol{\mu} \in \partial \mathbf{\Gamma}(\tilde{\boldsymbol{\zeta}})$. Thus, if (GOFP) has an optimal primal solution and dual solution, then these solutions are an equilibrium configuration of the dynamical network. To complete this part of the proof, it remains to show that $\tilde{\mathbf{y}}$ and $\tilde{\boldsymbol{\zeta}}$ are optimal solutions to (GOPP). Define $s(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\zeta}}) = \inf_{\mathbf{u},\boldsymbol{\mu}} \mathcal{L}(\mathbf{u}, \boldsymbol{\mu}, \tilde{\mathbf{y}})$ with $\tilde{\boldsymbol{\zeta}} = E^\top \tilde{\mathbf{y}}$. Clearly, $s(\tilde{\mathbf{y}}, \tilde{\boldsymbol{\zeta}}) = -\sum_{i=1}^{|\mathbf{V}|} K_i^\star(\tilde{y}_k) - \sum_{k=1}^{|\mathbf{E}|} \Gamma_k^{\star\star}(\tilde{\zeta}_k)$. Since $\Gamma_k^{\star\star} = \Gamma_k$ it can be readily seen that an optimal solution to $\inf_{\tilde{\mathbf{y}},\tilde{\boldsymbol{\zeta}}} s(\tilde{\mathbf{y}}\tilde{\boldsymbol{\zeta}})$ is an optimal solution to (GOPP). Thus, optimal solutions to (GOFP), (GOPP) are equilibrium configurations for the network. By the same argument follows that all possible network equilibrium configurations are solution to (GOFP), (GOPP).

It remains to prove convergence. Consider a network equilibrium configuration $\mathbf{u}$, $\mathbf{y}$, $\boldsymbol{\mu}$, and $\boldsymbol{\zeta}$. By assumption, the node dynamics are output strictly maximal equilibrium independent passive and since (GOFP), (GOPP) are feasible $\mathbf{u} \in \mathcal{U}$ and $\mathbf{y} \in \mathcal{Y}$. All controllers are maximal equilibrium independent passive and since (GOFP), (GOPP) are feasible, $\boldsymbol{\mu} \in \mathcal{M}$, and $\boldsymbol{\zeta} \in \mathcal{Z}$. Convergence of the trajectories follows now from the basic convergence result. □

6. Application: analysis of a traffic dynamics model

The potential of the proposed network optimization interpretation is now illustrated on the analysis of a nonlinear traffic dynamics models. The considered model is an *optimal velocity model*, as proposed in Bando, Hasebe, Nakayama, Shibata, and Sugiyama (1995) and Helbing and Tilch (1998), with the following assumptions: (i) the drivers are heterogeneous and have different "preferred" velocities, (ii) the influence between cars is bi-directional, and (iii) vehicles can overtake other vehicles. Each vehicle adjusts its velocity v_i according to

$$\dot{v}_i = \kappa_i[V_i(\Delta p) - v_i], \tag{25}$$

where $\kappa_i > 0$ is a constant and the adjustment $V_i(\Delta p)$ depends on the relative position to other vehicles, i.e., $\Delta p = p_j - p_i$, as

$$V_i(\Delta p) = V_i^0 + V_i^1 \sum_{j\in\mathcal{N}(i)} \tanh(p_j - p_i). \tag{26}$$

Here $\mathcal{N}(i)$ is used to denote the neighboring vehicles influencing vehicle i. Throughout this example we assume that the set of neighbors to a vehicle is not changing over time. The constants $V_i^0 > 0$ are "preferred velocities" and $V_i^1 > 0$ are "sensitivities" of the drivers. In the following we assume $V_i^0 \neq V_j^0$ (i.e., heterogeneity).

The model can be represented in the form (6), (7), (9), (22). The node dynamics can be identified as

$$\dot{v}_i(t) = \kappa_i[-v_i(t) + V_i^0 + V_i^1 u_i(t)], \quad y_i(t) = v_i(t), \tag{27}$$

with the velocity $v_i(t)$ being the node state. The input to each vehicle computes as $u_i(t) := \sum_{j\in\mathcal{N}(i)} \tanh(p_j(t) - p_i(t))$. The relative velocities of neighboring vehicles are $\boldsymbol{\zeta}(t) = E^\top \boldsymbol{y}$. Now, since $\dot{p}_i = v_i$, we can define the relative positions of neighboring vehicles as $\eta_k(t) = p_j(t) - p_i(t)$, where edge k connects nodes i and j. In vector notation, the coupling can be represented as

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\zeta}, \qquad \boldsymbol{\mu} = \tanh(\boldsymbol{\eta}), \tag{28}$$

and $\boldsymbol{u} = -E\boldsymbol{\mu}$, where $\tanh(\cdot)$ is here the vector valued function.

The node dynamics are output strictly maximal equilibrium independent passive systems. The equilibrium input–output map is the affine function $k_{y,i}(u_i) = V_i^0 + V_i^1 u_i$ and a corresponding storage function is $S_i = \frac{1}{2\kappa_i V_i^1}(v_i(t) - v_i)^2$, where v_i is the desired constant velocity. The objective functions associated to the node dynamics are the quadratic functions

$$K_i(u_i) = \frac{V_i^1}{2} u_i^2 + V_i^0 u_i \quad \text{and} \quad K_i^\star(y_i) = \frac{1}{2V_i^1}(y_i - V_i^0)^2. \tag{29}$$

Next, we show that the controller dynamics (28) is maximal equilibrium independent passive. Note that the output functions of (28) are monotone but bounded. The dynamics (28) will only attain a steady state for $\boldsymbol{\zeta} = 0$. However, if $\boldsymbol{\zeta} \neq 0$, the outputs will not grow unbounded, but will approach the saturation bounds of the nonlinearity. Thus, each of the coupling dynamics has the equilibrium input–output relation

$$\gamma_k(\zeta_k) = \begin{cases} +1 & \zeta_k > 0 \\ (-1, 1) & \zeta_k = 0 \\ -1 & \zeta_k < 0. \end{cases} \tag{30}$$

It can be easily verified that γ_k represents a maximal monotone relation in $\mathbb{R}^2$. To prove now maximal equilibrium independent passivity, we define the integral functions of the coupling nonlinearities, i.e., $P_k(\eta_k) = \ln\cosh(\eta_k)$. Note that the functions P_k are not strongly convex, as they asymptotically approach an affine function. The function P_k, the coupling nonlinearity $\psi_k(\eta_k) = \tanh(\eta_k)$, and the convex conjugate $P_k^\star(\mu_k)$ are illustrated in Fig. 3. The function P_k can now be used to prove maximal equilibrium independent passivity.

Proposition 6.1. *Each of the dynamics (28) is maximal equilibrium independent passive with equilibrium input–output relation (30).*

Proof. For any $\zeta_k = 0$ and any $\mu_k \in (-1, 1)$, there is a unique η_k such that $\mu_k = \tanh(\eta_k)$. The corresponding storage function

$$W_k(\eta_k(t)) = P_k(\eta_k(t)) - P_k(\eta_k) - \nabla P_k(\eta_k)(\eta(t) - \eta_k) \tag{31}$$

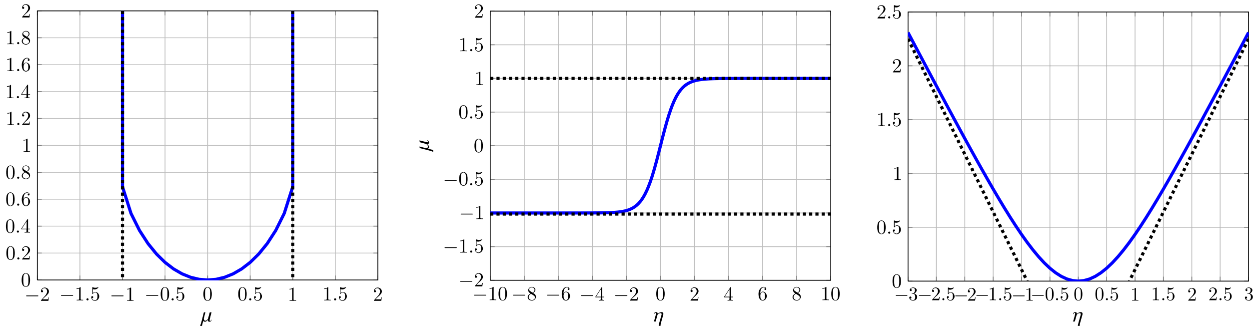
is positive definite. It can be readily seen that $\dot{W} = (\mu_k(t) - \mu_k)\zeta_k(t) = (\mu_k(t) - \mu_k)(\zeta_k(t) - \zeta_k)$. Furthermore, if $\zeta_r \neq 0$ we can define a sequence $\eta_k^1, \eta_k^2, \ldots$ that diverges to $+\infty$ if $\zeta_r > 0$ and to $-\infty$ if $\zeta_r < 0$. To each η_k^ℓ one can define the positive definite function (31), named $W_k^\ell(\eta_k)$. The sequence of functions W_k^ℓ approaches a positive semi-definite function $\bar{W}_k(\eta_k)$ that satisfies $\dot{\bar{W}}_k = (\mu_k(t) - \mu_k)\zeta_k(t)$. Additionally, we note that if $\zeta_k > 0$ ($\zeta_k < 0$) then $(\mu_k(t) - \mu_k) \leq 0$ ($(\mu_k(t) - \mu_k) \geq 0$) for all μ_k. Thus, it holds that $\zeta_k(\mu_k(t) - \mu_k) \leq 0$. Based on this observation we conclude $\dot{\bar{W}}_k \leq (\mu_k(t) - \mu_k)(\zeta_k(t) - \zeta_k)$. Thus, for each ζ_k and $\mu_k \in \gamma_k(\zeta_k)$, there exists a positive semi-definite storage function that allows to conclude passivity. □

To complete the network theoretic interpretation of the traffic dynamics model, we define the integral function of the input–output relation γ_k. The integral function of $\gamma_k(\zeta_k)$ is the absolute value of ζ_k and its convex conjugate is the indicator function for the set $[-1, 1]$, i.e.,

$$\Gamma_k(\zeta_k) = |\zeta_k|, \qquad \Gamma_k^\star(\mu_k) = I_{[-1,1]}(\mu_k).$$

Thus, for the traffic dynamics, the two network optimization problems (GOFP) and (GOPP) take a very characteristic structure. The optimal flow problem (GOFP) is almost identical to (OFP1), except that additionally constraints on the flow variables are imposed, i.e., the flows are constrained as $-1 \leq \mu_k \leq 1$. On the other hand, the optimal potential problem (GOPP) has a quadratic cost function for the potentials plus an additional absolute value of the potential differences, that can be understood as an ℓ_1-penalty.

Remark 6.2 (*Network Clustering*)**.** The connection of the presented results to the network clustering analysis presented in Bürger et al.



(a) $P^\star(\mu)$.

(b) $\mu = \nabla P(\eta)$ $(=: \psi(\eta))$.

(c) $P(\eta)$.

Fig. 3. Relation between the *flow cost function* $P^\star(\mu)$, the *coupling nonlinearity*, here $\psi(\eta) = \nabla P(\eta) := \tanh(\eta)$, and the *coupling function integral* $P(\eta)$.

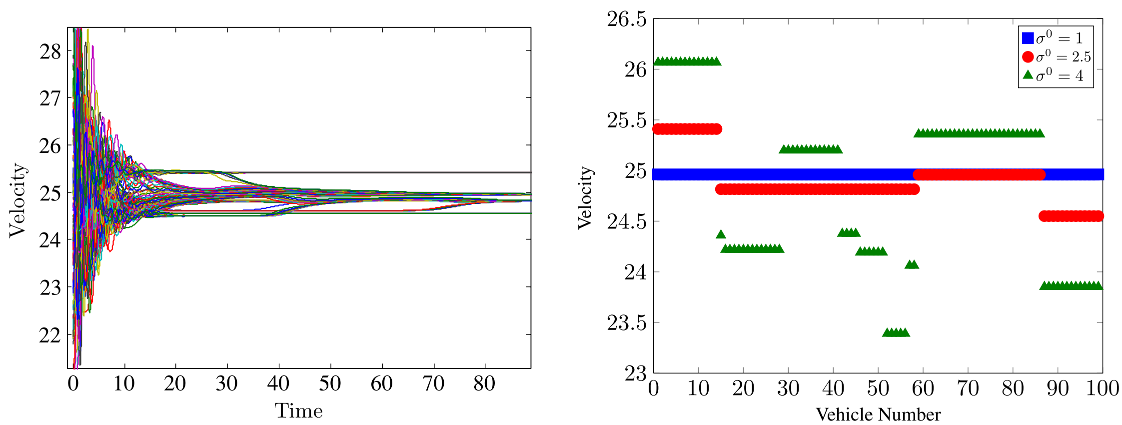


Fig. 4. Simulation results for a traffic dynamics model with 100 vehicles placed on a line graph. *Left:* Time trajectories of the velocities for normally distributed coefficients with $\sigma^0 = 2.5$ and $\sigma^1 = 1$. *Right:* Asymptotic velocities predicted by the network optimization problems for $\sigma^0 = 1$ (blue, '□'), $\sigma^0 = 2.5$ (red, 'o'), and $\sigma^0 = 4$ (green, 'Δ').

(2011) can be explained on the traffic dynamics. In Bürger et al. (2011) a *saddle-point problem* of the form

$$\max_{\substack{\mu_k \\ -1 \le \mu_k \le 1}} \min_{y_i} \mathcal{L}(\mathbf{y}, \boldsymbol{\mu}) := \sum_{i=1}^{|\mathbf{V}|} \frac{1}{2V_i^1}(y_i - V_i^0)^2 + \boldsymbol{\mu}^\top E^\top \mathbf{y}$$

is proposed to analyze and predict an asymptotic clustering behavior. Some straight forward manipulations reveal that the saddle-point problem results in fact from the Lagrange dual of (GOPP) for the traffic dynamics model. It has been shown in Bürger et al. (2011) that the solutions to the saddle-point problem eventually have a clustered structure.

We present a computational study with 100 vehicles placed on a line graph in Fig. 4. The sensitivity parameter is $\kappa = 0.6$ for all vehicles, while the parameters V_i^0 and V_i^1 are chosen as a common nominal parameter ($V_{nom}^0 = 25\frac{m}{s}$ and $V_1 = 10\frac{m}{s}$) plus a random component, i.e., $V_i^0 = V_{nom}^0 + V_{i,rand}^0$, chosen according to a zero mean normal distribution with different standard deviations. In Fig. 4 (left), the time-trajectories of the velocities v_i are shown with the random coefficients $V_{i,rand}^0$, $V_{i,rand}^1$ chosen from a distribution with $\sigma^0 = 2.5$ and $\sigma^1 = 1$, respectively. Fig. 4 (right) shows the asymptotic velocity distribution for different choices of the standard deviation σ^0. While for $\sigma^0 = 1$ the traffic agrees on a common velocity, already for $\sigma^0 = 2.5$ a clustering structure of the network can be seen. The clustering structure becomes more refined for $\sigma^0 = 4$. We have chosen for all studies $\sigma^1 = 1$. The network optimization framework provides us with efficient tools to analyze and predict the non-trivial asymptotic behavior, without the need to simulate the system for different parameter configurations.

7. Conclusions

We have established in this paper an intimate connection between passivity-based cooperative control and the network optimization theory of Rockafellar (1998). To obtain this connection, we introduced the notion of maximal equilibrium independent passivity as a variation of the equilibrium independent passivity concept of Hines et al. (2011). It was shown that dynamical networks involving maximal equilibrium independent passive systems asymptotically approach the solutions of several network optimization problems. For output agreement problems we have shown that the output agreement steady state is optimal with respect to an optimal flow and an optimal potential problem. This connection provided also an interpretation of the system outputs as potential variables, and of the system inputs as node divergence. Similar inverse optimality and duality results are established for general networks of maximal equilibrium independent passive systems. The general theory was illustrated on a nonlinear traffic dynamics model that shows asymptotically a clustering behavior. We believe that this result contributes to a unified understanding of networked dynamical systems and opens the way for further advanced analysis methods.

Acknowledgment

The authors thank the German Research Foundation (DFG) for financial support within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart.

References

Arcak, M. (2007). Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, *52*(8), 1380–1390.

Bai, H., Arcak, M., & Wen, J. (2011). *Cooperative control design: a systematic, passivity–based approach*. New York, NY: Springer.

Balakrishnan, V., & Vandenberghe, L. (1995). Connections between duality in control theory and convex optimization. In: *Proc. of the American control conference* (pp. 4030–4034).

Bando, M., Hasebe, K., Nakayama, A., Shibata, A., & Sugiyama, Y. (1995). Dynamical model of traffic congestion and numerical simulation. *Physical Review E*, *51*(2), 1035–1042.

Boyd, S., & Vandenberghe, L. (2003). *Convex optimization*. Cambridge: Cambridge University Press.

Bürger, M., Zelazo, D., & Allgöwer, F. (2011). Network clustering: a dynamical systems and saddle-point perspective. In: *Proc. of IEEE conference on decision and control*, Orlando, Florida, Dec. (pp. 7825–7830).

Bürger, M., Zelazo, D., & Allgöwer, F. (2012). Combinatorial insights and robustness analysis for clustering in dynamical networks. In: *Proc. of the American control conference*, Montreal (pp. 454–459).

Bürger, M., Zelazo, D., & Allgöwer, F. (2013). Hierarchical clustering of dynamical networks using a saddle-point analysis. *IEEE Transactions on Automatic Control*, *58*(1), 113–124.

Chopra, N., & Spong, M. (2006). *Advances in robot control, from everyday physics to human-like movements*. (pp. 107–134). Springer, ch. Passivity-based control of multi-agent systems.

Helbing, D., & Tilch, B. (1998). Generalized force model of traffic dynamics. *Physical Review E*, *58*, 133–138.

Hines, G. H., Arcak, M., & Packard, A. K. (2011). Equilibrium-independent passivity: a new definition and numerical certification. *Automatica*, *47*, 1949–1956.

Jayawardhana, B., Ortega, R., Gracia-Canseco, E., & Castanos, F. (2007). Passivity of nonlinear incremental systems: application to PI stabilization of nonlinear RLC circuits. *Systems & Control Letters*, *56*, 618–622.

Khalil, H. (2002). *Nonlinear systems*. Upper Saddle River, New Jersey: Prentice Hall.

Mesbahi, M., & Egerstedt, M. (2010). *Graph theoretic methods in multiagent networks*. Princeton University Press.

Rockafellar, R. (1997). *Convex analysis*. Princeton University Press.

Rockafellar, R. T. (1998). *Network flows and monotropic optimization*. Belmont, Massachusetts: Athena Scientific.

Scardovi, L., Arcak, M., & Sontag, E. D. (2010). Synchronization of interconnected systems with applications to biochemical networks: an input–output approach. *IEEE Transactions on Automatic Control*, *55*(6), 1367–1379.

Stan, G.-B., & Sepulchre, R. (2007). Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, *52*(2), 256–270.

Van der Schaft, A. J., & Maschke, B. M. (2013). Port–Hamiltonian systems on graphs. *SIAM Journal on Control and Optimization*, *51*, 906–937.

Zelazo, D., & Mesbahi, M. (2010). Edge agreement: graph-theoretic performance bounds and passivity analysis. *IEEE Transactions on Automatic Control*, *56*(3), 544–555.

Zhu, D., & Marcotte, P. (1995). New classes of generalized monotonicity. *Journal of Optimization Theory and Applications*, *87*(2), 457–471.

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